Quantum Numbers of Eigenstates of Generalized de Broglie-Bargmann-Wigner Equations for Fermions with Partonic Substructure

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Generalized de Broglie-Bargmann-Wigner (BBW) equations are relativistically invariant quantum mechanical many body equations with nontrivial interaction, selfregularization and probability interpretation. Owing to these properties these equations are a suitable means for describing relativistic bound states of fermions. In accordance with de Broglie's fusion theory and modern assumptions about the partonic substructure of elementary fermions, i.e., leptons and quarks, the three-body generalized BBW-equations are investigated. The transformation properties and quantum numbers of the three-parton equations under the relevant group actions are elaborated in detail. Section 3 deals with the action of the isospin group $SU(2)$, a $U(1)$ global gauge group for the fermion number, the hypercharge and charge generators. The resulting quantum numbers of the composite partonic systems can be adapted to those of the phenomenological particles to be described. The space-time transformations and in particular rotations generated by angular momentum operators are considered in Section 4. Based on the compatibility of the BBW-equations and the group theoretical constraints, in Sect. 5 integral equations are formulated in a representation with diagonal energy and total angular momentum variables. The paper provides new insight into the solution space and quantum labels of resulting integral equations for three parton states and prepares the ground for representing leptons and quarks as composite systems.

Key words: Relativistic Quantum Mechanics; Many Body Theory; Partonic Substructure of Leptons and Quarks; Group Theoretic Constraints.

1. Introduction

Generalized de Broglie-Bargmann-Wigner (BBW) equations are relativistically invariant quantum mechanical many body equations with nontrivial interaction, selfregularization and probability interpretation. Owing to these properties these equations are a suitable means for treating relativistic bound states of fermions.

The most simple problem in this formalism are the two body equations. This problem can be exactly solved and was treated in a preceding paper for the case of vector bosons, leading to the interpretation of the corresponding wave functions as a theoretical description of photons with partonic substructure, [1].

For the carrying out and the physical interpretation of such calculations an analysis of the symmetry properties, i.e., of the group structure of the solutions is essential. In particular a group theoretical analysis is imperative if one cannot exactly solve the corresponding equations. This is already the case in the three body (parton) problem which leads to integral equations of the Fredholm type. The latter equations are soluble in principle but hardly in practice. So for getting an information about the structure of the eigenvalue spectrum, the group theoretical analysis is the only means which allows to derive exact results concerning this spectrum. In the following we will discuss this problem.

In the original theory of de Broglie [2], and Bargmann and Wigner [3], the three body problem was extensively treated by Rarita and Schwinger [4] who concentrated on spin 3/2 solutions owing to a symmetry postulate on the spin part of the wave functions. Such a symmetry postulate on the spin part narrows down the manifold of solutions and is neither necessary in the original de Broglie's version nor for the generalized BBW-equations.

The group theoretical analysis of the generalized BBW-equations given here is based on the field theoretic properties of the three parton solutions and is intended to explore the parton structure of elementary fermions, i.e., of leptons and quarks. With respect to the latter species there are experimental signals that quarks are not elementary [5]. Then the most simple, nontrivial assumption consists in considering leptons and quarks as bound states of three partons. Such a hypothesis was inaugurated by Harari [6], and Shupe [7]. But apart from the assumption of the substructure to be given by three partons, our model has nothing in common with the Harari-Shupe model.

The fieldtheoretic background of our model was extensively discussed in [8, 9], so we refer for further information about the generalized BBW-equations and the corresponding model to these references.

In this paper we take these equations which result from this field theory for granted and give only a general discussion of their group theoretical properties. The physical interpretation of the corresponding solutions was already given in a preliminary way in preceding papers [9 – 12]. But it is the intention to improve these statements by a more stringent group theoretical analysis in forthcoming papers, based on the results obtained in this paper.

2. Relativistic Three-parton Equations

By means of the fieldtheoretic formalism wave equations for three-parton states can be derived [8, 9]. Here this extensive and comprehensive formalism into which these equations are incorporated cannot be described. Rather we concentrate on these equations themselves and their interpretation. For provisional guidance we assume that such equations and their states allow an appropriate description of leptons and quarks with partonic substructure. In this case the quantum numbers of those states must fit into the scheme of quantum numbers of the Standard model which is the topic of forthcoming papers, while in this paper the general group theoretical constraints will be discussed.

It is a pecularity of the field theoretic formalism that from the beginning this formalism is not specialized to any definite parton number n. And although we will exclusively deal with the parton number $n = 3$ in the following, the general field theoretic formulation is needed in order to be aware of the antisymmetry properties of the wave functions. Hence we start with the field theoretic version of the theory for hard core states which can be expressed by a single (covariant) functional equation. At this basic level of the theory it is convenient to use only symbolic general coordinate variables I which stand for the four dimensional space-time coordinate x and the algebraic indices Z. Then in this symbolic notation this hard core functional equation reads (using the summation convention), see [8, 9]:

$$K_{I_1 I}\partial_I|\mathcal{F}\rangle = U_{I_1 I_2 I_3 I_4}\big[F_{I_2 I} j_I \partial_{I_4}\partial_{I_3} + F_{I_3 I} j_I \partial_{I_2}\partial_{I_4} + F_{I_4 I} j_I \partial_{I_3}\partial_{I_2}\big]|\mathcal{F}\rangle. \quad (1)$$

Definitions of the various quantities which are contained in this symbolic equation will be given below. At first we explain the states $|\mathcal{F}\rangle$. These states are defined by

$$|\mathcal{F}(j)\rangle = \varphi_n(I_1 \dots I_n) j_{I_1} \dots j_{I_n}|0\rangle, \quad (2)$$

where φ_n is a formally normal ordered matrix element of the parton dynamics for hard core states, while the set of basis vectors $\{j_{I_1} \dots j_{I_n}|0\rangle\}$ is defined to be a fermionic Fock space with creation operators j_I and their duals ∂_K, which have not to be confused with ordinary particle creation and annihilation operators of quantum field theory as the former are elements of the generating functional space.

With regard to the application of equation (1) to the case $n = 3$, we choose in (2) the corresponding states and project (1) from the left hand side with $\langle 0|\partial_{N_1}\partial_{N_2}$. This yields

$$\sum_N K_{N_3 N}\mathcal{A}_{N_1 N_2 N}\varphi_{N_1 N_2 N} = \sum_{I_2 I_3 I_4} U_{N_3 I_2 I_3 I_4}\big[-3F_{I_2 N_2}\mathcal{A}_{N_1 I_3 I_4}\varphi_{N_1 I_3 I_4} + 3F_{I_2 N_1}\mathcal{A}_{N_2 I_3 I_4}\varphi_{N_2 I_3 I_4}\big]. \quad (3)$$

where the symbols $\mathcal{A}$ mean antisymmetrization in the corresponding indices. In all following calculations we omit the $\mathcal{A}$ symbols for brevity, but keep in mind that they are always present in the course of calculations. In order to perform such calculations one needs a more detailed representation of equations (3). For details of the evaluation of equations (3) in configuration space we refer to [8] and [9] as this evaluation is of no relevance with respect to the subsequent discussion. In particular we define the following quantities:

$\boldsymbol{r} \in R^3$, $x \in M^4$, and $Z = (i, \kappa, \alpha)$ where κ means superspin-isospin index, α = Dirac spinor index, i = auxiliary field index. The latter index characterizes the subfermion fields which are needed for the regularization procedure.

Let $\varphi_{Z_1Z_2Z_3}(x_1,x_2,x_3)$ be the covariant, antisymmetric state amplitude for the case $n=3$. Then from (3) the following equation can be derived for this state:

$$\big[D^{\mu}_{Z_3X_3}\partial_\mu(x_3)-m_{Z_3X_3}\big]\varphi_{Z_1Z_2X_3}(x_1,x_2,x_3)=$$
$$3U_{Z_3X_2X_3X_4}\big[-F_{X_2Z_2}(x_3-x_2)\varphi_{Z_1X_3X_4}(x_1,x_3,x_3)$$
$$+F_{X_2Z_1}(x_3-x_1)\varphi_{Z_2X_3X_4}(x_2,x_3,x_3)\big]. \quad (4)$$

Furthermore owing to the antisymmetrization in (3) one obtains two additional equations if the Dirac operator on the left hand side of (4) is applied to the coordinates x_1 and x_2. For brevity these two equations are not explicitly given, because apart from one exception, namely the derivation of the energy representation, these two equations are not needed if in every calculational step antisymmetrization is secured.

With respect to equation (4) the following definitions hold:

$$D^{\mu}_{Z_1Z_2}\ :=i\gamma^{\mu}_{\alpha_1\alpha_2}\delta_{\kappa_1\kappa_2}\delta_{i_1i_2} \quad (5)$$

and

$$m_{Z_1Z_2}\ :=m_{i_1}\delta_{\alpha_1\alpha_2}\delta_{\kappa_1\kappa_2}\delta_{i_1i_2} \quad (6)$$

and

$$F_{Z_1Z_2}(x_1-x_2)\ :=-i\lambda_{i_1}\delta_{i_1i_2}\gamma^5_{\kappa_1\kappa_2} \quad (7)$$
$$\cdot\,[(i\gamma^\mu\partial_\mu(x_1)+m_{i_1})C]_{\alpha_1\alpha_2}\Delta(x_1-x_2,m_{i_1}),$$

where $\Delta(x_1-x_2,m_{i_1})$ is the scalar Feynman propagator. The meaning of the index κ can be explained by decomposing it into two parts $\kappa\ :=(\Lambda,A)$ with $\Lambda=1,2$ superspin index of spinors and charge conjugated spinors and $A=1,2$ isospin index which can be equivalently expressed by $\kappa=1,2,3,4$.

The vertex term in equation (4) is fixed by the following definitions:

$$U_{Z_1Z_2Z_3Z_4}:=\lambda_{i_1}B_{i_2i_3i_4}V^{\kappa_1\kappa_2\kappa_3\kappa_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}, \quad (8)$$

where $B_{i_2i_3i_4}$ indicates the summation over the auxiliary field indices and where the vertex is given by a scalar and a pseudoscalar coupling of the subfermion fields

$$V^{\kappa_1\kappa_2\kappa_3\kappa_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}:=\frac{g}{2}\Big\{\big[\delta_{\alpha_1\alpha_2}C_{\alpha_3\alpha_4}-\gamma^5_{\alpha_1\alpha_2}(\gamma^5C)_{\alpha_3\alpha_4}\big] \quad (9)$$
$$\cdot\ \delta_{\kappa_1\kappa_2}\big[\gamma^5(1-\gamma^0)\big]_{\kappa_3\kappa_4}\Big\}_{as[2,3,4]}.$$

For vanishing coupling constant $g=0$ de Broglie's original fusion equations for local three fermion states are obtained, and for a solution of the whole set of equations only equation (4) has to be used, as for antisymmetric wave functions the remaining equations can be derived from (4) by interchange of indices. In this context it should be emphasized that the antisymmetry of wave functions is not an additional postulate. Rather it is an outcome of the general functional formalism which is used to derive such equations, see equation (3).

Concerning the physical interpretation of the wave functions it is closely related to the role of the auxiliary fields (indices) which appear in the corresponding equations and their solutions.

The task of the auxiliary fields is twofold: on the one hand they are used for regularization, on the other hand owing to their properties probability conservation can be deduced. As this topic was extensively treated for the two-parton case in [1] and the discussion of the three-parton case runs along the same lines we suppress the explicit deduction of these properties which are a special case of the general theory, see [13].

First we refer to the role of auxiliary fields in regularization, leading to the definition of the physical wave functions. We consider the wave functions of equation (4) with the full dependence on the auxiliary fields as unobservable, i.e., unphysical. In order to obtain the physical, singularity free wave functions in the case of three-parton states we decompose the index $Z:=(\alpha,\kappa,i)$ into $Z:=(z,i)$ and sum over i_1,i_2,i_3. This gives

$$\hat\varphi_{z_1z_2z_3}(x_1,x_2,x_3):=\sum_{i_1i_2i_3}\varphi_{Z_1Z_2Z_3}(x_1,x_2,x_3). \quad (10)$$

These functions are by definition the physical states. One immediately realizes that the physical wave function $\hat\varphi$ has the same transformation properties as the original wave function φ.

In order to derive a probability interpretation for the physical parton wave functions the single time formulation of (10) has to be used, see [8, 9] and in addition the single time energy equation has to be derived from (4), see [1]. Then with the single time density

$$\hat\varphi^\dagger\hat\varphi:=\sum_{z_1z_2z_3}\hat\varphi_{z_1z_2z_3}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,t)^* \quad (11)$$
$$\cdot\ \hat\varphi_{z_1z_2z_3}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,t)$$

for a general time dependent solution of the energy equation one obtains from this equation with $m_i = m + \delta m_i$, in the limit $\delta m_i \to 0$ current conservation:

$$\partial_t(\hat{\varphi}^\dagger \hat{\varphi}) + \sum_l \partial_k^l [\hat{\varphi}^\dagger \alpha^k(l) \hat{\varphi}] = 0. \tag{12}$$

This limit can be performed in the regularized wave functions without any difficulty after all calculations were done. Owing to current conservation the densities (11) are conserved positive quantities, i.e., the physical state amplitudes $\hat{\varphi}$ are elements of a corresponding Hilbert space with the norm expression (in the case under consideration!)

$$\langle\hat{\varphi}|\hat{\varphi}\rangle = \int \mathrm{d}^3 r_1 \mathrm{d}^3 r_2 \mathrm{d}^3 r_3 \hat{\varphi}_{z_1 z_2 z_3}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, t)^* \cdot \hat{\varphi}_{z_1 z_2 z_3}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, t), \tag{13}$$

and they describe the states of the system with interaction. Hence one is able to extract all quantum mechanically meaningful information about this system from its given state space.

Finally it should be noted that in the latter limit the coupling constants $\lambda_i g$ in the vertex (8), (9) of the three-parton equation diverge. But the essential point is that the regularized solutions of these equations remain finite in the whole range $(0, \infty)$ of $\lambda_i g$. Hence as the auxiliary fields are unobservable and the whole physics depends on the regularized solutions this behavior of the coupling constants has no obsevable consequences.

3. Algebraic Quantum Numbers

The transformation properties of the three parton wave functions are correlated to and determined by the transformation properties of the spinor field theory being the theoretical background for the derivation of the generalized BBW-equations.

The latter theory is formulated in terms of spinor fields $\psi_{\alpha Ai}(x)$ and formally charge conjugated spinor fields $\psi^c_{\alpha Ai}(x)$. For the definition of the indices see Section 2. In particular A is the index of a $SU(2)$ spinor basis. In this section we treat the algebraic quantum numbers representing the transformation properties of the three parton wave functions under these $SU(2)$ transformations and an additional $U(1)$ transformation.

The corresponding transformation matrices are given by, see [14]

$$U = \exp[-\frac{i}{2}\sum_{k=1}^{3} \varepsilon_k \sigma^k] \tag{14}$$

and lead to the transformed spinors

$$\psi'_{\alpha Ai}(x) = U_{AA'} \psi_{\alpha A' i}(x). \tag{15}$$

By definition the charge conjugated spinors are given by $\psi^c = C\bar{\psi}^T = C\gamma_0 \psi^*$. Then ψ^c transforms under $SU(2)$ transformations according to

$$\psi^c_{\alpha Ai}(x)' = U^*_{AA'} \psi^c_{\alpha A' i}(x) \tag{16}$$

with

$$U^* = \exp[-\frac{i}{2}\sum_{k=1}^{3} \varepsilon_k (-1)^k \sigma^k]. \tag{17}$$

We now combine spinors and charge conjugated spinors into a superspinor field by introducing the index κ, see Section 2. Then this superspinor field $\psi_{\alpha\kappa i}(x)$ transforms under $SU(2)$ transformations in the following way:

$$\psi'_{\alpha\kappa i}(x) = U_{\kappa\kappa'} \psi_{\alpha\kappa' i}(x) \tag{18}$$

with

$$U = \exp[-i\sum_{k=1}^{3} \varepsilon_k G^k], \tag{19}$$

where the superspin-isospin generators are given by

$$G^k_{Z_1 Z_2} = \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & (-)^k \sigma^k \end{pmatrix}_{\kappa_1 \kappa_2} \delta_{\alpha_1 \alpha_2}. \tag{20}$$

In addition the superspinors admit a $U(1)$ global gauge group with

$$U = \exp[-i\varepsilon F] \tag{21}$$

and

$$F_{Z_1 Z_2} = \frac{1}{3}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_{\kappa_1 \kappa_2} \delta_{\alpha_1 \alpha_2}. \tag{22}$$

Concerning the transformation properties of the three parton wave functions we consider for simplicity the physical wave functions $\hat{\varphi}$ in order to avoid the explicit dependence of the index set on the auxiliary field index i. The corresponding transformation properties are not changed by the transition from φ to $\hat{\varphi}$.

The transformation properties of the wave functions φ or $\hat{\varphi}$, respectively, must be compatible with the transformation properties of the spinor field theory in the background. For the global gauge groups this is the case if $\hat{\varphi}$ is transformed by

$$\hat{\varphi}^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)' = U_{\kappa_1\kappa'_1}U_{\kappa_2\kappa'_2}U_{\kappa_3\kappa'_3}\hat{\varphi}^{\kappa'_1\kappa'_2\kappa'_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3), \tag{23}$$

and of course this transformation property must be compatible with the generalized BBW-equations too.

But before demonstrating this compatibility we discuss the relation to the phenomenological quantum numbers. With the above choice of the symmetry group generators the elementary subfermions (partons) are associated to the isospin quantum number $t = 1/2$ and the fermion number $f = 1/3$. The fermion number of the subfermions (partons) can be arbitrarely chosen because the subfermions are unobservable. Once the subfermion quantum number is fixed (the isospin is treated in the conventional manner) there is no freedom for further manipulations. That means, charge and hypercharge for the subfermion bound states, i.e., for leptons and quarks are to be derived and have to coincide with the corresponding phenomenological values.

In order to reproduce the phenomenological charge we define a charge generator by

$$Q := G^3 + Y \tag{24}$$

with the hypercharge generator

$$Y := \frac{1}{2}F. \tag{25}$$

The corresponding quantum numbers are $q = t_3 + y$ and $y = f/2$ and the charge generator is explicitly given by

$$Q_{Z_1Z_2} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\kappa_1\kappa_2} \delta_{\alpha_1\alpha_2}. \tag{26}$$

If for brevity we introduce the general index $I = Z, x$, see Sect. 2, the isospin quantum numbers for the three-parton state are defined by the following conditions

$$\frac{9}{4}\varphi_{I_1I_2I_3} + 2\big[G^k_{I_1K_1}G^k_{I_2K_2}\varphi_{K_1K_2I_3} + G^k_{I_1K_1}G^k_{I_3K_2}\varphi_{K_1I_2K_2} + G^k_{I_2K_1}G^k_{I_3K_2}\varphi_{I_1K_1K_2}\big] = t(t+1)\varphi_{I_1I_2I_3} \tag{27}$$

$$G^3_{I_1K}\varphi_{KI_2I_3} + G^3_{I_2K}\varphi_{I_1KI_3} + G^3_{I_3K}\varphi_{I_1I_2K} = t_3\varphi_{I_1I_2I_3} \tag{28}$$

and the corresponding equations for the generators F, Y, and Q.

Concerning the compatibility of these transformations with the generalized BBW-equations, it is convenient to treat this problem by replacing equation (4) by the corresponding homogenous integral equation for bound states which reads:

$$\varphi_{Z_1Z_2Z_3}(x_1,x_2,x_3) = \int \mathrm{d}^4x G_{Z_3X_1}(x_3-x)U_{X_1X_2X_3X_4} \cdot 3\big[-F_{X_2Z_2}(x-x_2)\varphi_{Z_1X_3X_4}(x_1,x,x) + F_{X_2Z_1}(x-x_1)\varphi_{Z_2X_3X_4}(x_2,x,x)\big]. \tag{29}$$

Furthermore to simplify matters we sum in this equation over i_1, i_2, i_3, as the summation over auxiliary fields does not change the transformation properties of the wave function. Then one obtains with notation at full length

$$
\begin{aligned}
\hat\varphi^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = \frac{g}{2}\int \mathrm{d}^4x \sum\nolimits_i \lambda_i G_{\alpha_3\alpha_1'}(x_3-x,m_i)\delta_{\kappa_3\kappa_1'} \\
\cdot \sum\nolimits_h \Big\{ v^h_{\alpha_1'\beta}\delta_{\kappa_1'\rho}\big[(v^hC)_{\beta'\beta''}[\gamma^5(1-\gamma^0)]_{\rho'\rho''} - (v^hC)_{\beta''\beta'}[\gamma^5(1-\gamma^0)]_{\rho''\rho'}\big] \\
- v^h_{\alpha_1'\beta'}\delta_{\kappa_1'\rho'}\big[(v^hC)_{\beta\beta''}[\gamma^5(1-\gamma^0)]_{\rho\rho''} - (v^hC)_{\beta''\beta}[\gamma^5(1-\gamma^0)]_{\rho''\rho}\big] \\
- v^h_{\alpha_1'\beta''}\delta_{\kappa_1'\rho''}\big[(v^hC)_{\beta'\beta}[\gamma^5(1-\gamma^0)]_{\rho'\rho} - (v^hC)_{\beta\beta'}[\gamma^5(1-\gamma^0)]_{\rho\rho'}\big]\Big\} \\
\cdot\, 3\Big[-\sum\nolimits_j \lambda_j(-i)\gamma^5_{\rho\kappa_2}F_{\beta\alpha_2}(x-x_2,m_j)\hat\varphi^{\kappa_1\rho'\rho''}_{\alpha_1\beta'\beta''}(x_1,x,x) \\
+ \sum\nolimits_j \lambda_j(-i)\gamma^5_{\rho\kappa_1}F_{\beta\alpha_1}(x-x_1,m_j)\hat\varphi^{\kappa_2\rho'\rho''}_{\alpha_2\beta'\beta''}(x_2,x,x)\Big],
\end{aligned}
\tag{30}
$$

where $\sum_i \equiv \sum\limits_i$, etc., and the result of the following discussion can be summarized by

Proposition 1: The three parton generalized BBW-equations (30) are invariant under the global gauge group transformations (19), (21), i.e., with $\hat\varphi$ also $U\otimes U\otimes U\hat\varphi$ for any group element U are solutions of (30).

Proof: Equation (30) can be rewritten in the following form:

$$
\begin{aligned}
\hat\varphi^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = \int \mathrm{d}^4x \sum\nolimits_i \lambda_i G_{\alpha_3\eta}(x_3-x,m_i) \\
\cdot \sum\nolimits_h g\Big\{ v^h_{\eta\beta}(v^hC)_{\beta_1\beta_2}\gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2} 3\big[-\sum\nolimits_j \lambda_j(-i)F_{\beta_1\alpha_2}(x-x_2,m_j)\hat\varphi^{\kappa_1\rho_1\rho_2}_{\alpha_1\beta_1\beta_2}(x_1,x,x)\big] \\
+ v^h_{\eta\beta}(v^hC)_{\beta_1\beta_2}\gamma^5_{\kappa_3\kappa_1}\gamma^5_{\rho_1\rho_2} 3\big[\sum\nolimits_j \lambda_j(-i)F_{\beta_1\alpha_1}(x-x_1,m_j)\hat\varphi^{\kappa_2\rho_1\rho_2}_{\alpha_2\beta_1\beta_2}(x_2,x,x)\big] \\
- v^h_{\eta\beta_1}(v^hC)_{\beta\beta_2} 3\big[-\sum\nolimits_j \lambda_j(-i)F_{\beta\alpha_2}(x-x_2,m_j)\hat\varphi^{\kappa_1\kappa_3\kappa_2}_{\alpha_1\beta_1\beta_2}(x_1,x,x)\big] \\
- v^h_{\eta\beta_1}(v^hC)_{\beta\beta_2} 3\big[\sum\nolimits_j \lambda_j(-i)F_{\beta\alpha_1}(x-x_1,m_j)\hat\varphi^{\kappa_2\kappa_3\kappa_1}_{\alpha_2\beta_1\beta_2}(x_2,x,x)\big] \\
- v^h_{\eta\beta_2}(v^hC)_{\beta_1\beta} 3\big[-\sum\nolimits_j \lambda_j(-i)F_{\beta\alpha_2}(x-x_2,m_j)\hat\varphi^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\beta_1\beta_2}(x_1,x,x)\big] \\
- v^h_{\eta\beta_2}(v^hC)_{\beta_1\beta} 3\big[\sum\nolimits_j \lambda_j(-i)F_{\beta\alpha_1}(x-x_1,m_j)\hat\varphi^{\kappa_2\kappa_1\kappa_3}_{\alpha_2\beta_1\beta_2}(x_2,x,x)\big]\Big\}.
\end{aligned}
\tag{31}
$$

Applying $U\otimes U\otimes U$ to this equation it is obvious that in the last four terms of (31) this group operators directly act on $\hat\varphi$ on the right hand side of this equation. Hence we only have to study the application of $U\otimes U\otimes U$ to the first two terms on the right hand side of (31). As these terms have a similar structure it is sufficient to treat the first term only. Furthermore we can confine ourselves to discuss the action of infinitesimal transformations given by

$$
U(\epsilon_j) := 1 - i\epsilon_j G^j, \tag{32}
$$

and in the following formulas for brevity we suppress all indices and coordinates of the wave functions which are spectator indices, i.e., which are not involved in these transformations. Then one obtains

$$
\begin{aligned}
U(\epsilon_j)\otimes U(\epsilon_j)\otimes U(\epsilon_j)\hat\varphi &= \hat\varphi^{\kappa_1\kappa_2\kappa_3} - i\epsilon_j\big[G^j_{\kappa_1\kappa_1'}\hat\varphi^{\kappa_1'\kappa_2\kappa_3} + G^j_{\kappa_2\kappa_2'}\hat\varphi^{\kappa_1\kappa_2'\kappa_3} + G^j_{\kappa_3\kappa_3'}\hat\varphi^{\kappa_1\kappa_2\kappa_3'}\big] \\
&:= \hat\varphi^{\kappa_1\kappa_2\kappa_3} - i\epsilon_j \mathcal{G}^j_{\kappa_1\kappa_2\kappa_3,\kappa_1'\kappa_2'\kappa_3'}\hat\varphi^{\kappa_1'\kappa_2'\kappa_3'},
\end{aligned}
\tag{33}
$$

and we have to consider only that part of the first term which is directly involved in this transformation. This term reads

$$
\gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}\hat\varphi^{\kappa_1\rho_1\rho_2}_{\alpha_1\beta_1\beta_2}(x_1,x,x) \equiv \gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}\hat\varphi^{\kappa_1\rho_1\rho_2}. \tag{34}
$$

Application of $\mathcal{G}^j$ to this term gives

$$\mathcal{G}^j_{\kappa_1\kappa_2\kappa_3,\kappa_1'\kappa_2'\kappa_3'}\gamma^5_{\kappa_3'\kappa_2'}\gamma^5_{\rho_1\rho_2}\hat{\varphi}^{\kappa_1'\rho_1\rho_2}$$

$$= -i\epsilon_j\{\gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}G^j_{\kappa_1\kappa_1'}\hat{\varphi}^{\kappa_1'\rho_1\rho_2} + G^j_{\kappa_2\kappa_2'}\gamma^5_{\kappa_3\kappa_2'}\gamma^5_{\rho_1\rho_2}\hat{\varphi}^{\kappa_1\rho_1\rho_2} + G^j_{\kappa_3\kappa_3'}\gamma^5_{\kappa_3'\kappa_2}\gamma^5_{\rho_1\rho_2}\hat{\varphi}^{\kappa_1\rho_1\rho_2}\}. \quad (35)$$

Now by direct calculation one obtains

$$G^j_{\kappa_2\kappa_2'}\gamma^5_{\kappa_3\kappa_2'} + G^j_{\kappa_3\kappa_3'}\gamma^5_{\kappa_3'\kappa_2} = 0. \quad (36)$$

Hence (35) is equal to

$$\mathcal{G}^j_{\kappa_1\kappa_2\kappa_3,\kappa_1'\kappa_2'\kappa_3'}\gamma^5_{\kappa_3'\kappa_2'}\gamma^5_{\rho_1\rho_2}\hat{\varphi}^{\kappa_1'\rho_1\rho_2} = -i\epsilon_j\gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}G^j_{\kappa_1\kappa_1'}\hat{\varphi}^{\kappa_1'\rho_1\rho_2}. \quad (37)$$

On the other hand we consider the term

$$\gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}\mathcal{G}^j_{\kappa_1\rho_1\rho_2,\kappa_1'\rho_1'\rho_2'}\hat{\varphi}^{\kappa_1'\rho_1'\rho_2'}$$

$$= -i\epsilon_j\{\gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}G^j_{\kappa_1\kappa_1'}\hat{\varphi}^{\kappa_1'\rho_1\rho_2} + \gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}G^j_{\rho_1\rho_1'}\hat{\varphi}^{\kappa_1\rho_1'\rho_2} + \gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}G^j_{\rho_2\rho_2'}\hat{\varphi}^{\kappa_1\rho_1\rho_2'}\}. \quad (38)$$

The last two terms can be rearranged to give

$$\gamma^5_{\kappa_3\kappa_2}\left[\gamma^5_{\rho_2\rho_1}G^j_{\rho_1\rho_1'}\hat{\varphi}^{\kappa_1\rho_1'\rho_2} + \gamma^5_{\rho_1\rho_2}G^j_{\rho_2\rho_2'}\hat{\varphi}^{\kappa_1\rho_1\rho_2'}\right] = \gamma^5_{\kappa_3\kappa_2}[(\gamma^5G^j)_{\lambda\mu} + (\gamma^5G^j)_{\mu\lambda}]\hat{\varphi}^{\kappa_1\mu\lambda} = 0. \quad (39)$$

Hence

$$\mathcal{G}^j_{\kappa_1\kappa_2\kappa_3,\kappa_1'\kappa_2'\kappa_3'}\gamma^5_{\kappa_3'\kappa_2'}\gamma^5_{\rho_1\rho_2}\hat{\varphi}^{\kappa_1'\rho_1\rho_2} = \gamma^5_{\kappa_3\kappa_2}\gamma^5_{\rho_1\rho_2}\mathcal{G}^j_{\kappa_1\rho_1\rho_2,\kappa_1'\rho_1'\rho_2'}\hat{\varphi}^{\kappa_1'\rho_1'\rho_2'}. \quad (40)$$

Therefore $\mathcal{G}^j$ commutes with the first term on the right hand side of (31). In a similar way the second term can be treated. $\diamondsuit$

In order to discuss the symmetry properties of the solutions of equation (31), the most simple ansatz is given by

$$\hat{\varphi}^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = \Theta^l_{\kappa_1\kappa_2\kappa_3}\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) \quad (41)$$

and the requirement of Θ^l being symmetric in all indices under permutations. This ansatz only then leads to a simplified calculation if Θ^l can be eliminated from (31) and such a separation is only possible if Θ^l has no γ^5 contribution. Hence on Θ^l the additional condition

$$\gamma^5_{\kappa_1\kappa_2}\Theta^l_{\kappa_1\kappa_2\kappa_3} = 0 \quad (42)$$

must be imposed. Owing to this condition only sixteen of the twenty symmetric states Θ^l are admitted. These states are explicitly tabulated in [11] and a table with the physical interpretation of the quantum numbers is contained in [9]. But from this table it follows that the ansatz (41) is too simple to obtain a complete agreement with phenomenology. Such an agreement can be only achieved without the separation (41) and if (42) is replaced by the weaker condition

$$\gamma^5_{\kappa_1\kappa_2}\hat{\varphi}^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = 0. \quad (43)$$

The effect of this modification was discussed and tabulated in [12] by solving the corresponding energy equation in the strong coupling limit. It is our intention to treat this problem in the scope of BBW-equations in forthcoming papers based on the general group theoretical analysis given in this paper.

4. Angular Momentum Quantum Numbers

We directly discuss this problem by means of the covariant equation (29), because this equation is the basis for the whole formalism. For simplicity we postulate conditions (41) and (42) and as a consequence Θ^l can be eliminated from (29). As in the discussion of angular momentum the gauge group indices κ are only spectator indices this ansatz implies no loss of generality. After some rearrangements this leads to the following covariant equation

$$\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = 6g\int \mathrm{d}^4x\sum_i\lambda_iG_{\alpha_3\alpha_1'}(x_3-x,m_i) \cdot\sum_h\Big[v^h_{\alpha_1'\beta'}\hat{\varphi}_{\alpha_1\beta'\beta''}(x_1,x,x)(v^hC)_{\beta''\beta} \cdot\sum_j\lambda_j(i)F_{\beta\alpha_2}(x-x_2,m_j) - v^h_{\alpha_1'\beta'}\hat{\varphi}_{\alpha_2\beta'\beta''}(x_2,x,x)(v^hC)_{\beta''\beta} \cdot\sum_j\lambda_j(i)F_{\beta\alpha_1}(x-x_1,m_j)\Big]. \quad (44)$$

The behavior of the solutions of equation (44) under space-time transformations can be characterized by means of the Pauli-Lubanski spin vector W_μ. Owing to the relativistic invariance of (44) its solutions can be classified by the values of $W_\mu W^\mu$ and $P_\mu P^\mu$ as representations of the Poincare group. It is this property which constitutes the link to the quantum numbers of the energy equation. Owing to their transformation properties these solutions can be treated in the rest frame without loss of generality. In this case the Pauli-Lubanski spin vector reads

$$W_\mu = \frac{1}{2p_0}\varepsilon_{\mu\nu\rho 0} M^{\nu\mu} P^0, \tag{45}$$

and in this expression the representation of the generators P^μ and $M^{\mu\nu}$ depends on the dimension of the coordinate space. In the rest system W_0 vanishes and one obtains from (45)

$$W_i =_{\frac{1}{2}} \varepsilon_{ijk} M^{jk} = -J^i, \tag{46}$$

where the J^i are the angular momentum operators of the little group. Thus in the rest frame the quantum numbers of the solutions of equation (44) should be given by the eigenvalues of the Casimir operators of the little group $\mathbf{J}^2$ and J^3. In the following it will be demonstrated that equation (44) indeed is compatible with these group theoretic constraints, i.e., that the solutions can be classified as representations of this group.

Proposition 2: In the rest system of an eigenstate of equations (44) the eigenvalues of $\mathbf{J}^2$ and J^3 are good quantum numbers and their values are determined by those of the reduced solution $\hat\varphi(x_1, x, x)$.

Proof: The proof of this assertion depends crucially upon the transformation properties of the wavefunctions. In the spinor-charge conjugated spinor representation the wave functions transform as the direct product of Dirac spinors. This was shown in [15] and without further explanation we refer to [15]. Therefore in accordance with this transformation property in the three-fermion space the generators J^i are to be defined by

$$J^i = L^i + S^i = \sum_{\alpha=1,2,3} i\varepsilon_{ijk}[x_j^\alpha \partial_k^\alpha - x_k^\alpha \partial_j^\alpha] + \frac{1}{2}\left[\Sigma^i_{\alpha_1\alpha_1'}\delta_{\alpha_2\alpha_2'}\delta_{\alpha_3\alpha_3'} + \delta_{\alpha_1\alpha_1'}\Sigma^i_{\alpha_2\alpha_2'}\delta_{\alpha_3\alpha_3'} + \delta_{\alpha_1\alpha_1'}\delta_{\alpha_2\alpha_2'}\Sigma^i_{\alpha_3\alpha_3'}\right] \tag{47}$$

with $\partial_k^\alpha := \partial/\partial x^{k,\alpha}$. For brevity we discuss only the J^3 constraint because already by means of this condition the representations can be classified. In deducing this formula the strict distinction between co- and contravariant indices is imperative for the proof.

As in any solution procedure of equation (44) the antisymmetry of the wave function has to be secured, we antisymmetrize equation (44) explicitly and in addition substitute the corresponding vertex matrices in the resulting equation. This yields:

$$\hat\varphi_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)_{as} = \sum_{h_1h_2h_3}(-1)^P 2g\int d^4x \sum_i \lambda_i \int \frac{d^4p}{(2\pi)^4} f_i(\gamma^\mu p_\mu + m_i)_{\alpha_{h_1}\nu}\exp[-ip(x_{h_1}-x)] \cdot \{-\delta_{\nu\beta}\delta_{\nu'\beta''} + \gamma^5_{\nu\beta'}\gamma^5_{\nu'\beta''}\}\hat\varphi_{\alpha_{h_2}\beta'\beta''}(x_{h_2},x,x)\sum_j \lambda_j(i)\int\frac{d^4q}{(2\pi)^4} f_j[(-\gamma^\varrho q_\varrho + m_j)]_{\alpha_{h_3}\nu'}\exp[-iq(x-x_{h_3})]. \tag{48}$$

with $f_i := (p^2 - m_i^2)^{-1}$ and $f_j := (q^2 - m_j^2)^{-1}$.

To prepare the calculation of the J^3 constraint we apply finite spinor transformations $S(\Lambda)$ to equation (48) but keep by definition the space-time coordinates of the wave functions unchanged. This gives

$$S_{\varrho_1\alpha_1}S_{\varrho_2\alpha_2}S_{\varrho_3\alpha_3}\hat\varphi_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)_{as} = \sum_{h_1h_2h_3}(-1)^P 2g\int d^4x\sum_i\lambda_i\int\frac{d^4p}{(2\pi)^4} f_i S_{\varrho_{h_1}\alpha_{h_1}}(\gamma^\mu p_\mu + m_i)_{\alpha_{h_1}\alpha_1'} \cdot \exp[-ip(x_{h_1}-x)]\{-\delta_{\alpha_1'\beta'}S_{\varrho_{h_2}\alpha_{h_2}}\hat\varphi_{\alpha_{h_2}\beta'\beta''}(x_{h_2},x,x)\delta_{\beta''\beta} + \gamma^5_{\alpha_1'\beta'}S_{\varrho_{h_2}\alpha_{h_2}}\hat\varphi_{\alpha_{h_2}\beta'\beta''}(x_{h_2},x,x)\gamma^5_{\beta''\beta}\} \cdot \sum_j\lambda_j(i)\int\frac{d^4q}{(2\pi)^4} f_j S_{\varrho_{h_3}\alpha_{h_3}}(-\gamma^\varrho q_\varrho + m_j)_{\alpha_{h_3}\beta}\exp[-iq(x-x_{h_3})]. \tag{49}$$

Now we insert $S^{-1}S$ in all intermediate spin summations and make use of the transformation properties of the Dirac algebra. This induces coordinate transformations in (49) and transforms (49) into

$$S_{\varrho_1\alpha_1}S_{\varrho_2\alpha_2}S_{\varrho_3\alpha_3}\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = \sum_{h_1h_2h_3}(-1)^P 2g\int \mathrm{d}^4x\sum_i \lambda_i\int\frac{\mathrm{d}^4p'}{(2\pi)^4}f_i(\gamma^\nu p'_\nu+m_i)_{\varrho_{h_1}\nu}$$
$$\cdot\,\exp[-ip'_\nu\Lambda^\nu_\mu(x^\mu_{h_1}-x^\mu)]\{-\delta_{\nu\varphi}\delta_{\nu'\varphi'}S_{\varrho_{h_2}\alpha_{h_2}}S_{\varphi\beta'}S_{\varphi'\beta''}\hat{\varphi}_{\alpha_{h_2}\beta'\beta''}(x_{h_2},x,x)+\gamma^5_{\nu\varphi}\gamma^5_{\nu'\varphi'}S_{\varrho_{h_2}\alpha_{h_2}}$$
$$\cdot\,S_{\varphi\beta'}S_{\varphi'\beta''}\hat{\varphi}_{\alpha_{h_2}\beta'\beta''}(x_{h_2},x,x)\}\sum_j\lambda_j(i)\int\frac{\mathrm{d}^4q'}{(2\pi)^4}f_j(-\gamma^\kappa q'_\kappa+m_j)_{\varrho_{h_3}\nu'}\exp[-iq'_\kappa\Lambda^\kappa_\varrho(x^\varrho-x^\varrho_{h_3})]. \quad (50)$$

For infinitesimal rotations about the $\boldsymbol{e}_3$-axis the transformation matrices read

$$\Lambda^\nu_\mu = \begin{pmatrix} 1 & 0\\ 0 & \Lambda^l_j\end{pmatrix} \quad (51)$$

with, see [1, 16]

$$\Lambda^l_j=\delta^l_j-\epsilon(M_3)^l_j,\quad (\Lambda^{-1})^l_j=\delta^l_j+\epsilon(M_3)^l_j \quad (52)$$

and, see [17]

$$S(\Lambda)=\mathbf{1}-\frac{i}{2}\Sigma^3\epsilon,\quad S^{-1}(\Lambda)=\mathbf{1}+\frac{i}{2}\Sigma^3\epsilon. \quad (53)$$

If these infinitesimal transformations are substituted in equation (50) a power series expansion in ϵ gives in the lowest order ϵ^0 the original equation (48) which by definition is satisfied, while the first order term in ϵ^1 yields the following expression:

$$(-i)S^3_{\varrho_1\varrho_2\varrho_3,\alpha_1\alpha_2\alpha_3}\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = \sum_{h_1h_2h_3}(-1)^P 2g\int\mathrm{d}^4x\sum_i\lambda_i\int\frac{\mathrm{d}^4p'}{(2\pi)^4}f_i(\gamma^\nu p'_\nu+m_i)_{\varrho_{h_1}\nu}$$
$$\cdot\,\exp[-ip'_\nu(x^\nu_{h_1}-x^\nu)]\{[-\delta_{\nu\varphi}\delta_{\nu'\varphi'}+\gamma^5_{\nu\varphi}\gamma^5_{\nu'\varphi'}][(-i)S^3_{\varrho_{h_2}\varphi\varphi',\alpha_{h_2}\beta'\beta''}+\delta_{\varrho_{h_2}\alpha_{h_2}}\delta_{\varphi\beta'}\delta_{\varphi'\beta''}p'_l(i)(M_3)^l_j(x^j_{h_1}-x^j)$$
$$+\,\delta_{\varrho_{h_2}\alpha_{h_2}}\delta_{\varphi\beta'}\delta_{\varphi'\beta''}q'_l(i)(M_3)^l_j(x^j-x^j_{h_3})]\hat{\varphi}_{\alpha_{h_2}\beta'\beta''}(x_{h_2},x,x)\}$$
$$\cdot\sum_j\lambda_j(i)\int\frac{\mathrm{d}^4q'}{(2\pi)^4}f_j(-\gamma^\kappa q'_\kappa+m_j)_{\varrho_{h_3}\nu'}\exp[-iq'_\kappa(x^\kappa-x^\kappa_{h_3})]. \quad (54)$$

If afterwards L^3 is applied to equation (48), the subsequent addition of $i\times$(54) and this L^3-term gives with (47) the final formula:

$$J^3_{\varrho_1\varrho_2\varrho_3,\alpha_1\alpha_2\alpha_3}\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = \sum_{h_1h_2h_3}(-1)^P 2g\int\mathrm{d}^4x\int\frac{\mathrm{d}^4p}{(2\pi)^4}\sum_i f_i(\gamma^\nu p_\nu+m_i)_{\varrho_{h_1}\nu}\exp[-ip_\nu(x^\nu_{h_1}-x^\nu)]$$
$$\cdot\{[-\delta_{\nu\varphi}\delta_{\nu'\varphi'}+\gamma^5_{\nu\varphi}\gamma^5_{\nu'\varphi'}][S^3_{\varrho_{h_2}\varphi\varphi',\alpha_{h_2}\beta'\beta''}+\delta_{\varrho_{h_2}\alpha_{h_2}}\delta_{\varphi\beta'}\delta_{\varphi'\beta''}L^1_{h_2}+\delta_{\varrho_{h_2}\alpha_{h_2}}\delta_{\varphi\beta'}\delta_{\varphi'\beta''}L^1_x]\hat{\varphi}_{\alpha_{h_2}\beta'\beta''}(x_{h_2},x,x)\}$$
$$\cdot\sum_j\lambda_j(i)\int\frac{\mathrm{d}^4q}{(2\pi)^4}f_j(-\gamma^\kappa q_\kappa+m_j)_{\varrho_{h_3}\nu'}\exp[-iq_\kappa(x^\kappa-x^\kappa_{h_3})], \quad (55)$$

where the primes of p and q were omitted and by the admixture of the L^3-terms the unwanted terms in (54) cancel out.

Let us now assume that

$$[S^3_{\varrho_h\varphi\varphi',\alpha_h\beta'\beta''}+\delta_{\varrho_h\alpha_h}\delta_{\varphi\beta'}\delta_{\varphi'\beta''}(L^1_h+L^1_x)]\hat{\varphi}_{\alpha_h\beta'\beta''}(x_h,x,x)=j_3\hat{\varphi}_{\varrho_h\varphi\varphi'}(x_h,x,x) \quad (56)$$

holds. Then by substitution of (56) into (55) and comparison with (48) one obtains

$$J^3_{\varrho_1\varrho_2\varrho_3,\alpha_1\alpha_2\alpha_3}\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3)_{as}=j^3\hat{\varphi}_{\varrho_1\varrho_2\varrho_3}(x_1,x_2,x_3)_{as}. \quad (57)$$

Hence by (57) it follows that the J^3 constraint is satisfied and that the corresponding eigenvalue results from equation (56). $\diamondsuit$

We assume that an analogous result holds for the $\mathbf{J}^2$ condition without doing the rather complicated calculations explicitly. Then the existence of these constraints in addition to the energy eigenvalue equation means: The state space can be decomposed into a set of irreducible representation spaces of the little group. In analogy to the two-parton case we assume that only the lowest dimensional representations lead to stable bound states, i.e., we concentrate on the discussion of these representations. The most simple lowest dimensional representation is a spin 1/2 representation with orbital angular momentum zero. This representation describes a composite spin 1/2 fermion which in combination with the superspin-isospin quantum numbers we identify with the members of the lepton generations. The next higher dimensional representation contains an orbital angular momentum 1 and a spin angular momentum 1/2. The former leads to a triplett which is energetically degenerate. This triplett forms a representation of the little group rotations, i. e., an $O(3)$ representation and, in combination with the other quantum numbers, should be identified with the quark generations, see [8 – 10].

5. Eigenstates of Energy and Angular Momentum

Owing to the translational invariance of the three-parton equation their solutions admit a representation where the total four momentum is diagonalized. This leads to the ansatz

$$\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = \exp\Big[-ik(x_1+x_2+x_3)\frac{1}{3}\Big] \cdot \hat{\chi}_{\alpha_1\alpha_2\alpha_3}(x_2-x_1, x_3-x_2). \tag{58}$$

In the rest system $k=(k_0,0,0,0)$ the total angular momentum commutes with the translational part of (58), i.e.

$$[\exp[-ik(x_1+x_2+x_3)]_{/k=k_0}, J_3]_- = 0 \tag{59}$$

and therefore in this system the energy eigenvalue and the angular momentum eigenvalue can be simultaneously calculated.

First we discuss the energy eigenvalue equation. We define:

$$F_1(x)_{\alpha\beta} = \sum_i \lambda_i \int \frac{\mathrm{d}^4p}{(2\pi)^4} f_i(\gamma^\mu p_\mu + m_i)_{\alpha\beta} \cdot \exp[-ip_\kappa x^\kappa] \tag{60}$$

and

$$F_2(x)_{\alpha\beta} = \sum_j \lambda_j(i) \int \frac{\mathrm{d}^4q}{(2\pi)^4} f_j(-\gamma^\nu q_\nu + m_j)_{\alpha\beta} \cdot \exp[-iq_\chi x^\chi] \tag{61}$$

and

$$V_{\alpha\beta\gamma\delta} = [-\delta_{\alpha\beta}\delta_{\gamma\delta} + \gamma^5_{\alpha\beta}\gamma^5_{\gamma\delta}]. \tag{62}$$

Using these definitions equation (48) can be rewritten in the following form:

$$\begin{aligned}
\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_1,x_2,x_3) = 2g\Big\{ &\int \mathrm{d}^4x \hat{F}_1(x_1-x)_{\alpha_1\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_2\beta\beta'}(x_2,x,x) \hat{F}_2(x-x_3)_{\alpha_3\nu'} \\
&- \int \mathrm{d}^4x \hat{F}_1(x_2-x)_{\alpha_2\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_1\beta\beta'}(x_1,x,x) \hat{F}_2(x-x_3)_{\alpha_3\nu'} \\
&+ \int \mathrm{d}^4x \hat{F}_1(x_2-x)_{\alpha_2\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_3\beta\beta'}(x_3,x,x) \hat{F}_2(x-x_1)_{\alpha_1\nu'} \\
&- \int \mathrm{d}^4x \hat{F}_1(x_3-x)_{\alpha_3\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_2\beta\beta'}(x_2,x,x) \hat{F}_2(x-x_1)_{\alpha_1\nu'} \\
&+ \int \mathrm{d}^4x \hat{F}_1(x_3-x)_{\alpha_3\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_1\beta\beta'}(x_2,x,x) \hat{F}_2(x-x_2)_{\alpha_2\nu'} \\
&- \int \mathrm{d}^4x \hat{F}_1(x_1-x)_{\alpha_1\nu} V_{\nu\beta\nu'\beta'} \hat{\varphi}_{\alpha_3\beta\beta'}(x_3,x,x) \hat{F}_2(x-x_2)_{\alpha_2\nu'} \Big\},
\end{aligned} \tag{63}$$

where the permutations over h_1, h_2, h_3 in (48) are written in full.

Substitution of (58) into (63) and transformation by

$$z = \frac{1}{3}(x_1 + x_2 + x_3); \quad u = (x_2 - x_1); \quad v = (x_3 - x_2) \tag{64}$$

then leads with $x_2 = x_3$ or $v = 0$ after elimination of the translational part to the following equation

$$\begin{aligned}
\hat{\chi}_{\alpha_1\alpha_2\alpha_3}(u,0) = 2g \int \mathrm{d}^4x V_{\nu\beta\nu'\beta'} \\
\cdot \Big\{ & \hat{F}_1(z - \frac{2}{3}u - x)_{\alpha_1\nu} \exp[-ik(\frac{1}{3}u + 2x)\frac{1}{3}]\hat{\chi}_{\alpha_2\beta\beta'}(x - z - \frac{1}{3}u, 0)\hat{F}_2(x - z - \frac{1}{3}u)_{\alpha_3\nu'} \\
- & \hat{F}_1(z + \frac{1}{3}u - x)_{\alpha_2\nu} \exp[-ik(-\frac{2}{3}u + 2x)\frac{1}{3}]\hat{\chi}_{\alpha_1\beta\beta'}(x - z + \frac{2}{3}u, 0)\hat{F}_2(x - z - \frac{1}{3}u)_{\alpha_3\nu'} \\
+ & \hat{F}_1(z + \frac{1}{3}u - x)_{\alpha_2\nu} \exp[-ik(\frac{1}{3}u + 2x)\frac{1}{3}]\hat{\chi}_{\alpha_3\beta\beta'}(x - z - \frac{1}{3}u, 0)\hat{F}_2(x - z + \frac{2}{3}u)_{\alpha_1\nu'} \\
- & \hat{F}_1(z + \frac{1}{3}u - x)_{\alpha_3\nu} \exp[-ik(\frac{1}{3}u + 2x)\frac{1}{3}]\hat{\chi}_{\alpha_2\beta\beta'}(x - z - \frac{1}{3}u, 0)\hat{F}_2(x - z + \frac{2}{3}u)_{\alpha_1\nu'} \\
+ & \hat{F}_1(z + \frac{1}{3}u - x)_{\alpha_3\nu} \exp[-ik(-\frac{2}{3}u + 2x)\frac{1}{3}]\hat{\chi}_{\alpha_1\beta\beta'}(x - z + \frac{2}{3}u, 0)\hat{F}_2(x - z - \frac{1}{3}u)_{\alpha_2\nu'} \\
- & \hat{F}_1(z - \frac{2}{3}u - x)_{\alpha_1\nu} \exp[-ik(\frac{1}{3}u + 2x)\frac{1}{3}]\hat{\chi}_{\alpha_3\beta\beta'}(x - z - \frac{1}{3}u, 0)\hat{F}_2(x - z - \frac{1}{3}u)_{\alpha_2\nu'} \Big\}.
\end{aligned} \tag{65}$$

With the translation $x = x' + z - \frac{2}{3}u$, or $x = x' + z + \frac{1}{3}u$, respectively, one obtains the final form of the energy equation:

$$\begin{aligned}
\hat{\chi}_{\alpha_1\alpha_2\alpha_3}(u,0) = 2g \int \mathrm{d}^4x' V_{\nu\beta\nu'\beta'} \\
\cdot \Big\{ & \hat{F}_1(-u - x')_{\alpha_1\nu} \exp[-ik(-2u + 2x')\frac{1}{3}]\hat{F}_2(x')_{\alpha_3\nu'}\hat{\chi}_{\alpha_2\beta\beta'}(x', 0) \\
- & \hat{F}_1(u - x')_{\alpha_2\nu} \exp[-ik(-2u + 2x')\frac{1}{3}]\hat{F}_2(x' - u)_{\alpha_3\nu'}\hat{\chi}_{\alpha_1\beta\beta'}(x', 0) \\
+ & \hat{F}_1(-x')_{\alpha_2\nu} \exp[-ik(u + 2x')\frac{1}{3}]\hat{F}_2(x' + u)_{\alpha_3\nu'}\hat{\chi}_{\alpha_3\beta\beta'}(x', 0) \\
- & \hat{F}_1(-x')_{\alpha_3\nu} \exp[-ik(u + 2x')\frac{1}{3}]\hat{F}_2(x' + u)_{\alpha_1\nu'}\hat{\chi}_{\alpha_2\beta\beta'}(x', 0) \\
+ & \hat{F}_1(u - x')_{\alpha_3\nu} \exp[-ik(-2u + 2x')\frac{1}{3}]\hat{F}_2(x' - u)_{\alpha_2\nu'}\hat{\chi}_{\alpha_1\beta\beta'}(x', 0) \\
- & \hat{F}_1(-u - x')_{\alpha_1\nu} \exp[-ik(u + 2x')\frac{1}{3}]\hat{F}_2(x')_{\alpha_2\nu'}\hat{\chi}_{\alpha_3\beta\beta'}(x', 0) \Big\}.
\end{aligned} \tag{66}$$

Obviously equation (66) is a selfconsistent integral equation for the reduced function $\hat{\chi}(u,0)$. As in equation (63) the antisymmetry of the full solution is guaranteed by construction, the resulting equations (65) or (66), respectively, must fulfil those conditions which are a consequence of the antisymmetry of this full solution for the reduced function $\hat{\chi}(u,0)$.

Suppose now that $\hat{\chi}(u,0)$ is a solution of equation (66). Then using this solution one can calculate $\hat{\varphi}(x_l, x, x),\ l = 1,2,3$ by means of equation (58). In particular one obtains from (58)

$$\hat{\varphi}_{\alpha_1\alpha_2\alpha_3}(x_l, x, x) = \exp[-ik(x_l + 2x)\frac{1}{3}]\hat{\chi}_{\alpha_1\alpha_2\alpha_3}(x - x_l, 0) \tag{67}$$

and substituting these functions into (63) only by integrations the full solutions can be generated.

Equation (66) is a rather complicated integral equation. Thus one cannot expect to derive exact solutions although in principle such solutions must exist according to the Fredholm theory. On the other hand we know that the full equation (63) is compatible with the angular momentum constraint (57). Hence any solution of (63) must fulfil (57). This in its turn is guaranteed if the angular momentum condition (56) is satisfied. Hence for a first classification of the energy spectrum of equation (63) we analyze condition (56) in the rest system. Substitution of (58) into (56) then gives after elimination of the center of mass term

$$S^3_{\rho\varphi\varphi',\alpha\beta\beta'} + \delta_{\rho\alpha}\delta_{\varphi\beta}\delta_{\varphi'\beta'}(L^3_l + L^3_x)]\hat{\chi}_{\alpha\beta\beta'}(x - x_l, 0)$$
$$= j_3\hat{\chi}_{\rho\varphi\varphi'}(x - x_l, 0). \quad (68)$$

We now apply the transformation $z = (x_l + 2x)\frac{1}{3}$, $u_l = (x - x_l)$ and this transformation leads to $L^3_l + L^3_x = L^3_z + L^3_{u_l}$ which yields for (68) the final formula

$$[S^3_{\rho\varphi\varphi',\alpha\beta\beta'} + \delta_{\rho\alpha}\delta_{\varphi\beta}\delta_{\varphi'\beta'}L^3_{u_l}]\hat{\chi}_{\alpha\beta\beta'}(u_l, 0)$$
$$= j_3\hat{\chi}_{\rho\varphi\varphi'}(u_l, 0). \quad (69)$$

Of course the argument u_l can be replaced by u as for all u_l eqution (69) is referred to the same function $\hat{\chi}$ and the same angular quantum number j_3.

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